IV. *A general Method for exhibiting the* Value *of an* Algebraic Expreſſion *involving ſeveral* Radical Quantities *in an Infinite Series: Wherein Sir* Iſaac Newton'*s Theorem for involving a* Binomial, *with another of the ſame Author, relating to the Roots of Equations, are demonſtrated. By* T. Simpſon *F. R. S.*

Read Jan. 10. 1750-1.

AMONG all the great improvements, which the art of computation hath in theſe laſt ages received, the method of ſeries may be juſtly one of the moſt conſiderable; ſince not only the doctrine of chances and annuites, with ſome other branches of the mathematics, depend almoſt intirely thereon, but even the buſineſs of fluents, of ſuch extenſive uſe, would, without its aid and concurrence, be quite at a ſtand in a multitude of caſes, as is well known to mathematicians.

It is for this reaſon, that the celebrated binomial theorem, for converting radical quantities into ſeries's, is ranked, by many, among the principal diſcoveries of its illuſtrious author; ſeeing, by means thereof, a vaſt number of fluents are found, that would otherwiſe be impracticable: nor is there any caſe, however complex, to which it may not be extended.

It is true, when two or more compound radical quantities are involved together, the operation, by having two or more ſeries's to multiply into one another,

another, becomes very troublesome and laborious; and, what is worse, the Law of continuation, whereby a part of the labour might be avoided, is exceedingly hard, if not impossibe, this way to be discovered. In the following paper something is attempted towards obviating the said inconveniencies; but whether the success has been answerable, I shall not take upon me to determine.

Problem I.

To find a series exhibiting the value of $\overline{1+\frac{x}{a}}\big|^{m} \times \overline{1+\frac{x}{b}}\big|^{n} \times \overline{1+\frac{x}{c}}\big|^{p} \times \overline{1+\frac{x}{d}}\big|^{q}$ &c. in simple terms; x being indeterminate, and a, b, c, d, m, n, p, &c. any given numbers, whole or broken, positive or negative.

Put $u=\overline{1+\frac{x}{a}}\big|^{m}$, $w=\overline{1+\frac{x}{b}}\big|^{n}$, $y=\overline{1+\frac{x}{c}}\big|^{p}$, $z=1+\frac{x}{d}^{q}$ &c.

Also let $\Delta = uwyz$, &c. (= the quantity proposed)

Then, in fluxions $\dot{\Delta} = \dot{u}wyz$, &c. $+ u\dot{w}yz$, &c. $+ uw\dot{y}z$, &c. $+ uwy\dot{z}$, &c &c. Which equation, divided by the preceding one, gives $\frac{\dot{\Delta}}{\Delta} = \frac{\dot{u}}{u} + \frac{\dot{w}}{w} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z}$ &c.

But,

But, since $u = \overline{1 + \frac{x}{a}}\Big|^{m}$, we have $\dot{u} = m\dot{x} \times \overline{1 + \frac{x}{a}}\Big|^{m-1}$;

and therefore $\frac{\dot{u}}{u} = \frac{m\dot{x}}{a} \times \overline{1 + \frac{x}{a}}\Big|^{-1} = \frac{m\dot{x}}{a} \times \overline{1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \frac{x^4}{a^4}\ \&c.}$ by Division.

And in the same manner it appears, that $\frac{\dot{w}}{w} = \frac{n\dot{x}}{b} \times \overline{1 - \frac{x}{b} + \frac{x}{b}\ \&c.}\ \&c.$

Hence, our equation, by substituting these values, becomes

$$\frac{\dot{\Delta}}{\Delta} = \dot{x} \times \left\{ \begin{array}{l} \frac{m}{a} - \frac{mx}{a^2} + \frac{mx^2}{a^3} - \frac{mx^3}{a^4}\ \&c. \\ \frac{n}{b} - \frac{nx}{b^2} + \frac{nx^2}{b^3} - \frac{nx^3}{b^4}\ \&c. \\ \frac{p}{c} - \frac{px}{c^2} + \frac{px^2}{c^3} - \frac{px^3}{c^4}\ \&c. \\ \&c. \quad \&c. \quad \&c. \quad \&c. \end{array} \right\}$$

Put $P = \frac{m}{a} + \frac{n}{b} + \frac{p}{c} + \frac{q}{d}\ \&c.$

$Q = \frac{m}{a^2} + \frac{n}{b^2} + \frac{p}{c^2} + \frac{q}{d^2}\ \&c.$

$R = \frac{m}{a^3} + \frac{n}{b^3} + \frac{p}{c^3} + \frac{q}{d^3}\ \&c.$

&c. &c.

Then it will be

$$\frac{\dot{\Delta}}{\Delta} = \dot{x} \times \overline{P - Qx + Rx^2 - Sx^3 + Tz^4 - Vx^5\ \&c.}$$

Assume

Assume $\Delta = A + Bx + Cx^2 + Dx^3 + Ex^4$, &c. let this value, with that of $\dot{\Delta}$, be substituted in the last equation: from whence, by comparing the homologous terms, there will come out

$B = PA$

$C = \frac{PB - QA}{2}$

$D = \frac{PC - QB - RA}{3}$

$E = \frac{PD - QC + RB - SA}{4}$

$F = \frac{PE - QD + RC - SB + TA}{5}$

$G = \frac{PF - QE + RD - SC + TB - VA}{6}$

&c.

Where the law of continuation is manifest, and where it is also evident, that the value of (A) the first term of the required series, must be an unit; because, when $x = 0$, then the given expression becomes $1^m \times 1^n \times 1^p = 1$. Q. E. I.

Corol. 1.

If a be taken $= 1$, and n, p, q, &c. each $= 0$; then will $P = m$, $Q = m$, $R = m$, &c. And therefore

$A = 1$

$B = m$

$2C = mB - mA$

$3\mathcal{D} = mC - mB + mA = mC - 2C$
$4E = m\mathcal{D} - mC + mB - mA = m\mathcal{D} - 3\mathcal{D}$
&c.

Consequently $C = \frac{m.\overline{m-1}}{2}$, $\mathcal{D} = \frac{C \times \overline{m-2}}{3} = \frac{m.\overline{m-1}.\overline{m-2}}{2.\ 3}$, $E = \frac{E \times \overline{m-3}}{4} = \frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{2.3.4}$ &c.

Hence, in this case, $1 + mx + \frac{m.\overline{m-1}}{2}x^2 + \frac{m.\overline{m-1}.\overline{m-2}}{2.3}x^3$ &c. $(= A + Bx + Cx^2$ &c.$) = \overline{1+x}|^m$: which series is the same with that given by Sir Isaac Newton.

Corol. 2.

If a be taken $= \frac{1}{a}$, $\beta = \frac{1}{b}$, $\gamma = \frac{1}{c}$, &c. and $z = \frac{1}{x}$ then will the proposed expression be transformed to $\overline{1+\frac{\alpha}{z}}|^m \times \overline{1+\frac{\beta}{z}}|^n \times \overline{1+\frac{\gamma}{z}}|^p \times \overline{1+\frac{\delta}{z}}|^q$ &c.

Also $\mathcal{P} = m\alpha + n\beta + p\gamma +$ &c.
$\mathcal{Q} = m\alpha^2 + n\beta^2 + p\gamma^2 +$ &c.
$R = m\alpha^3 + n\beta^3 + p\gamma^3 +$ &c.
&c.

And consequently $\overline{1+\frac{\alpha}{z}}|^m \times \overline{1+\frac{\beta}{z}}|^n \times \overline{1+\frac{\gamma}{z}}|^p \times \overline{1+\frac{\delta}{z}}|^q$ &c.

&c. $= A + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z^3}$ &c. where $A = 1$, $B = PA$, $C = \frac{PB - QA}{2}$ &c. as before. Which equation or theorem anſwers in caſe of a deſcending ſeries.

Corol. 3.

Hence, if each of the quantities m, n, p, &c. be taken equal to unity, and their number be denoted by v; then will $\overline{1 + \frac{\alpha}{z}} \times \overline{1 + \frac{\beta}{z}} \times \overline{1 + \frac{\gamma}{z}} \times \overline{1 + \frac{\delta}{z}}$ &c. $= A + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z^3}$ &c. Which equation, multiplied by z^v, gives $\overline{z+\alpha} \times \overline{z+\beta} \times \overline{z+\gamma} \times \overline{z+\delta}$ &c. $= Az^v + Bz^{v-1} + Cz^{v-2} + Dz^{v-3}$ &c.

Whence it appears, that $\overline{z-\alpha} \times \overline{z-\beta} \times \overline{z-\gamma} \times \overline{z-\delta}$ &c. is $= Az^v - Bz^{v-1} + Cz^{v-2} - Dz^{v-3}$ &c. Where $A = 1$, $B = PA$, $C = \frac{PB - QA}{2}$, $D = \frac{PC - QB + RA}{3}$, &c. (as before); P being in this caſe = ſum of all the quantities α, β, γ, δ, &c. Q = the ſum of all their ſquares; R = the ſum of their cubes, &c. &c.

Corol. 4.

Since α, β, γ, δ, &c. are the roots of the equation, $z^v - Bz^{v-1} + Cx^{v-2} - Dz^{v-3}$, &c. $= 0$; it

follows,

follows, that, if B, C, D, E, &c. be given; the ſum of thoſe roots (P); the ſum of their ſquares (Q), and the ſum of their cubes (R) &c. will alſo be given from the foregoing equations: whence will be had

$$\begin{aligned}
P &= B\\
Q &= +PB-2C\\
R &= -PC+QB+3D\\
S &= +PD-QC+RB-4E\\
T &= -PE+QD-RC+SB+5F\\
&\text{\&c.} \qquad \text{\&c.}
\end{aligned}$$

where the law of continuation is obvious.

Theſe values are the ſame with thoſe given (without demonſtration) by Sir Iſaac Newton, in his Univerſal Arithmetic, for finding when ſome of the roots of an equation are impoſſible.

PROBLEM II.

To find a ſeries expreſſing the value of $\overline{1+\frac{x}{a}}\Big|^{m}$

$$\times\ \overline{1+\frac{x^2}{b}}\Big|^{n} \times \overline{1+\frac{x^3}{c}}\Big|^{p} \times \overline{1+\frac{x^4}{d}}\Big|^{q}, \text{ \&c.}$$

By putting $u=\overline{1+\frac{x}{a}}\Big|^{m}$, $w=\overline{1+\frac{x^2}{b}}\Big|^{n}$, &c.; and proceeding as in the laſt problem; there will be had

$$\frac{\dot{u}}{u}=\frac{m\dot{x}}{a}\times\overline{1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}\text{ \&c.}}$$

$$\frac{\dot{w}}{w}=\frac{2uxx\dot{x}}{b}\times\overline{1-\frac{x^2}{b}+\frac{x^4}{b^2}-\frac{x^6}{b^3}\text{ \&c.}}$$

$$\text{\&c.} \qquad \text{\&c.}$$

Whence,

Whence, making $P=\frac{m}{a}$, $Q=\frac{m}{a^2}-\frac{2n}{b}$, $R=\frac{m}{a^3}+\frac{3p}{c}$, $S=\frac{m}{a^4}+\frac{2n}{b^2}-\frac{4q}{d}$, $T=\frac{m}{a^5}+\frac{5r}{e}$, *&c.* and assuming $A+Bx+Cx^2+Dx^3+Ex^4$, *&c.* to express the series sought, the several values of A, B, C, D, *&c.* will be exhibited by the very equations brought out in the resolution of the preceding problem.

V. *A Letter from* George Bayly *M. D. of* Chichester, *to* Henry Pemberton *M. D. F. R. S. &c. of the Use of the* Bark *in the* Small-Pox.

Dear Sir,

Read Jan. 10. 1750.

THE case I lately mention'd to you in conversation, of which you desired a more particular account, is, as far as I have been able to recollect at this distance of time, as follows.

The patient, a gentlewoman of a fat corpulent habit, and healthy constitution, but 73 years of age, was, on the 6 day of December 1742, seiz'd with the common symptoms of a fever, attended with a sudden great loss of strength; so that, being carried to bed, she was not able to sit upright in it for the least space of time, without being held up by her assistant.